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The Orthogonality Postulate in Axiomatic Quantum Mechanics

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Abstract

Let $p(A,\alpha,E)$ be the probability that a measurement of an observable A for the system in a state α will lead to a value in a Borel set E. An experimental function is a function f from the set of all states $\mathscr{S}$ into $[0,1]$ for which there are an observable A and a Borel set E such that $f(\alpha) = p(A,\alpha,E)$ for all $\alpha \in \mathscr{S}$. A sequence $f_1, f_2, \ldots$ of experimental functions is said to be orthogonal if there is an experimental function g such that $g + f_1 + f_2 + \ldots = 1$, and it is said to be pairwise orthogonal if $f_i + f_j \leqslant 1$ for $i \neq j$. It is shown that if we assume both notions to be equivalent then the set L of all experimental functions is an orthocomplemented partially ordered set with respect to the natural order of real functions with the complementation $f' = 1 - f$, each observable A can be identified with an L-valued measure μ_A, each state α can be identified with a probability measure m_α on L and we have $p(A,\alpha,E) = m_\alpha \circ \mu A(E)$. Thus we obtain the abstract setting of axiomatic quantum mechanics as a consequence of a single postulate.

1. *Introduction*

Following Mackey (1963) we assume that with each quantum mechanical system F we can associate the set of all observables $\mathscr{O}$, the set of all states $\mathscr{S}$, and a function $p: \mathscr{O} \times \mathscr{S} \times \mathscr{B}(R) \to [0,1]$, where $\mathscr{B}(R)$ is the set of all Borel subsets of the real line R. The function $p(A,\alpha,E)$ is interpreted as the probability that a measurement of the observable A for the system in the state will lead to a value in the Borel set E. Consequently, for each fixed A and each fixed α the map $E \to p(A,\alpha,E)$ is a probability measure on $\mathscr{B}(R)$. The function p with this property will be called the probability function of our system. The aim of a physical theory is to throw $p(A,\alpha,E)$ into a concrete form that will allow us to calculate the values of p and the mean value of A in every state α. In quantum mechanics an observable A determines a projection-valued measure P^A in a Hilbert space $\mathscr{H}$ and thus by the spectral theorem a self-adjoint operator (which is denoted also by A), every (pure) state can be identified with a unit vector in $\mathscr{H}$, and the values of p can be calculated from the formula

$$p(A, \alpha, E) = (P_E{}^A \alpha, \alpha)$$

The mean value of A in the state α is then

$$\exp_\alpha(A) = (A\alpha, \alpha)$$

As an intermediate step, before making the assumption about the existence of the Hilbert space $\mathscr{H}$ associated with p we can assume that there is an orthocomplemented partially ordered set L (called the logic of p) such that each observable A corresponds to an L-valued measure μ_A, each state α corresponds to a probability measure m_α on L, and $p(A, \alpha, E) = m_\alpha \circ \mu_A(E)$ for all $E \in \mathscr{B}(R)$. To pass to the Hilbert space formalism it suffices to assume that L is isomorphic to the orthocomplemented partially ordered set of orthogonal projections on a Hilbert space $\mathscr{H}$. On the other hand, if we assume that L is the Boolean algebra of the Borel subsets of the phase space for our system (postulating that the phase space exists), we get the formalism of classical statistical mechanics. It turns out that several important theorems about observables and states in quantum mechanics can be proved already in this abstract setting of the orthocomplemented partially ordered set L without involving the Hilbert space $\mathscr{H}$. Moreover, an abstract system of observables and states associated with L is interesting on its own and can be investigated from a pure mathematical point of view. The theory developed then is frequently called axiomatic quantum mechanics. A full exposition of the present state of axiomatic quantum mechanics can be found in a paper of Gudder (1970).

Mackey (1963) showed that the existence of an orthocomplemented partially ordered set associated with the probability function p can be deduced from a set of seven assumptions (called axioms) which admit some physical interpretation. In the present paper we would like to show that the existence of L is in fact a consequence of a single postulate which admits a very simple interpretation. This postulate is strongly related to the orthogonality properties of L and indicates on orthogonality as a basic notion in any physical theory.

2. *The Orthogonality Postulate*

Before we state our main theorem we need to introduce some auxiliary notions.

Let $\mathscr{O}$ and $\mathscr{S}$ be respectively the set of all observables and the set of all states of a system F and let $p: \mathscr{O} \times \mathscr{S} \times \mathscr{B}(R) \to [0,1]$ be the probability function for F. We may form the set of all pairs (A, E) where $A \in \mathscr{O}$ and $E \in \mathscr{B}(R)$. Every pair (A, E) will be called an experimental pair. In the set $\mathscr{E}$ of all experimental pairs we introduce an equivalence relation defined by $(A_1, E_1) \sim (A_2, E_2)$ if and only if for every $\alpha \in \mathscr{S}$ $p(A_1, \alpha, E_1) = p(A_2, \alpha, E_2)$.

The set of all equivalence classes $L_0 = \mathscr{E}/\sim$ will be called the logic of p, and the members of L_0, denoted by $|(A,E)|$, will be called experimental propositions. Every experimental proposition $|(A,E)|$ uniquely determines a real-valued function $f_{A,E}$ from the set of all states $\mathscr{S}$ into $[0,1]$ defined by $f_{A,E}(\alpha) = p(A,\alpha,E)$ for all $\alpha \in \mathscr{S}$. In fact, we have $(A_1,E_1) \sim (A_2,E_2)$ if and only if $f_{A_1,E_1} = f_{A_2,E_2}$. The set of all such functions will be denoted by L, i.e. $L = \{f_{A,E}: A \in \mathscr{O}, E \in \mathscr{B}(R)\}$, and members of L will be called experimental functions induced by p. We see that there is a one-to-one correspondence between experimental propositions and experimental functions, and consequently L may also be called the logic of p. If f is an experimental function corresponding to the experimental proposition $e = |(A,E)|$, then the value of f at each $\alpha \in \mathscr{S}$, $f(\alpha)$, can be interpreted as the probability that e will turn out to be true in the state α (i.e. the probability that a measurement of A in the state α will lead to a value in E).

Experimental functions are real-valued functions and we can operate on them as on real functions. If $f, g \in L$, then $f+g$ and $f-g$ denote respectively the functions $f(x)+g(x)$ and $f(x)-g(x)$, $f \leqslant g$ means that $f(x) \leqslant g(x)$ for all $x \in \mathscr{S}$. We can also consider infinite sums of experimental functions $f_1 + f_2 + \ldots$; namely, $g = f_1 + f_2 + \ldots$ means that the series $f_1(\alpha) + f_2(\alpha) + \ldots$ is convergent for each $\alpha \in \mathscr{S}$ and $g(\alpha) = f_1(\alpha) + f_2(\alpha) + \ldots$. 0 and 1 will denote the functions (with domain $\mathscr{S}$) equal to 0 and to 1, respectively.

Definition 1. A sequence (finite or countable) of experimental functions $f_1, f_2, \ldots \in L$, is said to be orthogonal if there is an experimental function $g \in L$ such that $g + f_1 + f_2 + \ldots = 1$.

The reason for the use of this terminology will become clear in the sequel where we show that a sequence orthogonal in the sense of Definition 1 is orthogonal in the usual sense as a subset of some orthocomplemented partially ordered set. At this moment, however, the orthogonality is to be understood only formally as defined above.

Definition 2. A sequence (finite or countable) of experimental functions $f_1, f_2, \ldots \in L$, is said to be pairwise orthogonal if $f_i + f_j \leqslant 1$ for $i \neq j$, $i, j = 1, 2, \ldots$.

A one-element sequence is by definition pairwise orthogonal.

Observe that an orthogonal sequence is pairwise orthogonal but in general not conversely. If in an orthogonal sequence $f_1, f_2, \ldots$ one function, say f_i, takes the value 1 at some $\alpha \in \mathscr{S}$, $f_i(\alpha) = 1$, then all the other functions necessarily take the value 0 at this, i.e. $f_j(\alpha) = 0$ for $j \neq i$. We obtain exactly the same conclusion for a pairwise orthogonal sequence. Let e_1, $e_2, \ldots$ be the experimental propositions corresponding to an orthogonal sequence of experimental functions $f_1, f_2, \ldots$. If some proposition e_i has the probability 1 in a state α, then all the others have probability 0 in this state. Loosely speaking this means that if the proposition e_i is true in the state α, then all the others are false in that state. We get exactly the same interpretation for a sequence of experimental propositions whose corresponding sequence of experimental function is pairwise orthogonal. We see that in this interpretation both notions (orthogonality and pairwise orthogonality) are equivalent. This of course does not mean that both definitions are equivalent. However, the discussion above strongly supports the hypothesis that it is reasonable to assume that both notions are equivalent in any physical theory. We will show that such an assumption is equivalent to assuming that the set of experimental functions (and consequently the set of experimental propositions) is in a natural way an orthocomplemented partially ordered set. This means that both notions are equivalent, at least in such theories as classical mechanics and quantum mechanics. Moreover, this assumption agrees with the general understanding of orthogonality in mathematics, where orthogonality is always understood as pairwise orthogonality (independently of how this last term is defined). Accordingly, we make the following very plausible postulate.

The Orthogonality Postulate. A sequence of experimental functions is orthogonal if and only if it is pairwise orthogonal.

We shall now investigate the consequences of this postulate.

Theorem. Let $p\colon \mathcal{O} \times \mathcal{S} \times \mathcal{B}(R) \to [0,1]$ be a probability function for which the orthogonality postulate holds. Then the set L of experimental functions induced by p (the logic of p) is an orthocomplemented partially ordered set with respect to the natural order of real functions ($f \leqslant g$ if and only if $f(x) \leqslant g(x)$ for all $x \in \mathcal{S}$) with the complementation $f' = 1 - f$. Each observable $A \in \mathcal{O}$ determines a unique L-valued measure $\mu_A\colon \mathcal{B}(R) \to L$ defined by $\mu_A(E) = f_{A,E}$ for all $E \in \mathcal{B}(R)$, and each state α determines a unique probability measure m_α on L $m_\alpha\colon L \to [0,1]$ defined by $m_\alpha(f) = f(\alpha)$ for all $f \in L$. The family of L-valued measures corresponding to all observables $\{\mu_A\colon A \in \mathcal{O}\}$ is surjective, and the family of probability measures corresponding to all states $\{m_\alpha\colon \alpha \in \mathcal{S}\}$ is full. For each $A \in \mathcal{O}$, each $\alpha \in S$, and each $E \in \mathcal{B}(R)$ we have

$$p(A, \alpha, E) = m_\alpha \circ \mu_A(E)$$

Conversely, if L is an arbitrary orthocomplemented partially ordered set admitting a full set of probability measures $\mathcal{S}$, and $\mathcal{O}$ is a surjective set of L-valued measures, then the function p from $\mathcal{O} \times \mathcal{S} \times \mathcal{B}(R)$ into $[0,1]$ defined by $p(A,\alpha,E) = \alpha \circ A(E)$ for all $A \in \mathcal{O}$, $\alpha \in \mathcal{S}$, $E \in \mathcal{B}(R)$, is a probability function satisfying the orthogonality postulate and the logic of p is isomorphic to L.

We see that the notion of an orthocomplemented partially ordered set admitting a full set of probability measures is equivalent to the notion of a probability function satisfying the orthogonality postulate. Since we are interested only in calculating the values of p, observables of a physical system can be identified with L-valued measures and states with probability measures on L for a suitable L.

Before we prove our theorem we shall recall the definitions of the notions appearing in it. The terminology is consistent with Mackey (1963).

A partially ordered set $(L, \leqslant)$ is said to be orthocomplemented if there is a mapping $'$ of L into L such that

1⁰ $f'' = f$ for all $f \in L$.

2⁰ $f \leqslant g$ implies $g' \leqslant f'$ for all $f, g \in L$.

3⁰ If $f_1, f_2, \ldots$ is a sequence of members of L for which $f_i \leqslant f_j'$ for $i \neq j$, then the least upper bound $f_1 \cup f_2 \cup \ldots$ exists in $(L, \leqslant)$.

4⁰ $f \cup f' = g \cup g'$ for all $f, g \in L$ ($f \cup f'$ will be denoted by 1).

5⁰ $f \leqslant g$ implies $g = f \cup (f \cup g')'$.

In a lattice property 5⁰ is equivalent to orthomodularity (see Maeda & Maeda, 1970). Accordingly, a partially ordered set satisfying conditions 1⁰–5⁰ should be called an orthomodular σ-orthocomplemented partially ordered set. For simplicity, however, we stick to the terminology used by Mackey.

We denote $f \leqslant g'$ by $f \perp g$.

Let L be an orthocomplemented partially ordered set. A mapping $\mu: \mathscr{B}(R) \to L$ is said to be an L-valued measure if $E \cap F = \varnothing$ implies $\mu(E) \perp \mu(F)$ and $\mu(E_1 \cup E_2 \cup \ldots) = \mu(E_1) \cup \mu(E_2) \cup \ldots$ whenever $E_i \cap E_j = \varnothing$ for $i \neq j$, $i, j = 1, 2, \ldots$. A family $\{\mu_t : t \in T\}$ of L-valued measures is said to be surjective if for each $f \in L$ there are $t \in T$ and $E \in \mathscr{B}(R)$ such that $\mu_t(E) = f$.

A mapping $m: L \to [0,1]$ is said to be a probability measure on L if $m(1) = 1$ and $m(f_1 \cup f_2 \cup \ldots) = m(f_1) + m(f_2) + \ldots$ whenever $f_i \perp f_j$ for $i \neq j$, $i, j = 1, 2, \ldots$. A family $\{m_\alpha : \alpha \in \mathscr{S}\}$ of probability measures on L is said to be full if $m_\alpha(f) \leqslant m_\alpha(g)$ for all $\alpha \in \mathscr{S}$ implies $f \leqslant g$. Note that not every orthomodular poset admits a full set of probability measures. There are examples of orthomodular partially ordered sets (even lattices) which admit no probability measures at all (Meyer, 1970; Greechie, 1971).

Proof of the Theorem. We first show that the orthogonality postulate implies that L has the following three properties:

(i) The zero function belongs to L.

(ii) $f \in L$ implies $1 - f \in L$.

(iii) For any sequence $f_1, f_2, \ldots$ of members of L satisfying $f_i + f_j \leqslant 1$ for $i \neq j$ we have $f_1 + f_2 + \ldots \in L$.

In fact, since a one-element sequence is by definition pairwise orthogonal, by the orthogonality postulate it is orthogonal; that is, for each $f \in L$ there is $g \in L$ such that $f + g = 1$. Consequently, $1 - f$ belongs to L and (ii) holds. If $f \in L$, then the sequence f, $1 - f$ is pairwise orthogonal, consequently it is orthogonal and there is $g \in L$ such that $g + f + (1 - f) = 1$. This implies that $g \equiv 0$ belongs to L and (i) holds. Finally, a sequence $f_1, f_2, \ldots$ satisfying $f_i + f_j \leqslant 1$ for $i \neq j$ is orthogonal by the postulate and there is $g \in L$ such that $g + f_1 + f_2 + \ldots = 1$. This implies $f_1 + f_2 + \ldots = 1 - g \in L$. Hence (iii) also holds.

We now make L into a partially ordered set by defining $f \leqslant g$ if and only if $f(\alpha) \leqslant g(\alpha)$ for all $\alpha \in \mathscr{S}$, and define a map $'$ of L into L by $f' = 1 - f$ for all $f \in L$. It is evident that $(L, \leqslant, ')$ satisfies conditions 1^0 and 2^0. We shall prove 3^0. Observe first that taking in (iii) $f_{n+1} = f_{n+2} = \ldots = 0$ we see that for any finite sequence $f_1, f_2, \ldots, f_n$ satisfying $f_i + f_j \leqslant 1$ for $i \neq j$ we have $f_1 + f_2 + \ldots + f_n \in L$. We have in L $f_1 \perp f_2$ equivalent to $f_1 + f_2 \leqslant 1$. We first prove that for $f_1 \perp f_2$, $f_1 \cup f_2$ exists and $f_1 \cup f_2 = f_1 + f_2$. We have $f = f_1 + f_2 \in L$. Clearly $f_1 \leqslant f$ and $f_2 \leqslant f$. Let $g \in L$, $f_1 \leqslant g$ and $f_2 \leqslant g$. This means that $f_1 + g' \leqslant 1$ and $f_2 + g' \leqslant 1$. Hence the sequence f_1, f_2, g', 0, 0,… satisfies the assumption in (iii) and consequently $f_1 + f_2 + g' \in L$, which implies $f_1 + f_2 + g' \leqslant 1$, i.e. $f_1 + f_2 \leqslant g$. Thus $f \leqslant g$, which implies $f_1 \cup f_2 = f$. We now proceed by induction. Assume that for any sequence of length n $f_1, f_2, \ldots, f_n, f_i \in L$, satisfying $f_i + f_j \leqslant 1$ for $i \neq j$, $f_1 \cup f_2 \cup \ldots \cup f_n$ exists and $f_1 \cup f_2 \cup \ldots \cup f_n = f_1 + f_2 + \ldots + f_n$. Let $f_1, f_2, \ldots, f_n, f_{n+1}$ be any sequence of members of L where $f_i + f_j \leqslant 1$ for $i \neq j$. By (iii) we infer that $f_1 + f_2 + \ldots + f_{n+1} \in L$. By the induction hypothesis $f = f_1 + f_2 + \ldots + f_n = f_1 \cup f_2 \cup \ldots \cup f_n$. Consequently $f + f_{n+1} \leqslant 1$. By the part just proved, $f \cup f_{n+1} = f + f_{n+1}$. Hence $f_1 \cup f_2 \cup \ldots \cup f_{n+1} = f_1 + f_2 + \ldots + f_{n+1}$. Now let $f_1, f_2, \ldots$ be a sequence of members of L where $f_i + f_j \leqslant 1$ for $i \neq j$. By (iii) we have $f = f_1 + f_2 + \ldots \in L$. Clearly $f_i \leqslant f$. We must show that $f = f_1 \cup f_2 \cup \ldots$. Let $f_i \leqslant g$, $i = 1, 2, \ldots$ for some $g \in L$. Then $f_1 \cup f_2 \cup \ldots \cup f_n$ exists for $n = 1, 2, \ldots$ and $f_1 \cup f_2 \cup \ldots \cup f_n = f_1 + f_2 + \ldots + f_n$. Consequently $f_1 + f_2 + \ldots + f_n \leqslant g$ for $n = 1, 2, \ldots$. Hence $f_1 + f_2 + \ldots \leqslant g$, i.e. $f \leqslant g$. This shows that $f = f_1 \cup f_2 \cup \ldots$ exists and equals $f_1 + f_2 + \ldots$. Hence 3^0 holds. For any $f \in L$ we have $f + (1 - f) \leqslant 1$, i.e. $f \perp f'$. By the part just proved, $f \cup f'$ exists in L and $f \cup f' = f + f' = 1 + (1 - f) = 1$. So 4^0 holds. To show that 5^0 also holds, let $f \leqslant g$, $f, g \in L$. This implies that $f + (1 - g) \leqslant 1$ and $f \cup g' = f + g' = f + (1 - g)$. Consequently, $h = (f \cup g')' = 1 - (1 + f - g) = g - f \in L$. Hence $f + h = g \leqslant 1$. Hence $f \perp h$ and $f \cup h = f + h$. We see that $f \cup (f \cup g')' = f + (g - f) = g$, which means that 5^0 holds. Thus $(L, \leqslant, ')$ is an orthocomplemented partially ordered set.

For each $\alpha \in \mathscr{S}$, the mapping m_α of L into $[0,1]$ is a probability measure. We have by definition $m_\alpha(f) = f(\alpha)$ for all $\alpha \in \mathscr{S}$. Consequently, $m_\alpha(1) = 1(\alpha) = 1$, and for $f_1, f_2, \ldots \in L$, with $f_i \perp f_j$ for $i \neq j$ the least upper bound $h = f_1 \cup f_2 \cup \ldots$ exists and $h = f_1 + f_2 + \ldots$. This implies $m_\alpha(f_1 \cup f_2 \cup \ldots) = m_\alpha(h) = h(\alpha) = f_1(\alpha) + f_2(\alpha) + \ldots = m_\alpha(f_1) + m_\alpha(f_2) + \ldots$, which shows that m_α is a probability measure on L. The family $\{m_\alpha : \alpha \in \mathscr{S}\}$ is full, since $m_\alpha(f) \leqslant m_\alpha(g)$ for all $\alpha \in \mathscr{S}$ means that $f(\alpha) \leqslant g(\alpha)$ for all $\alpha \in \mathscr{S}$, which coincides with the definition of the order in L.

For each $A \in \mathscr{O}$, the mapping μ_A of $\mathscr{B}(R)$ into L defined by $\mu_A(E) = f_{A,E}$ for all $E \in \mathscr{B}(R)$ is an L-valued measure. In fact, $E \cap F = \varnothing$ implies $p(A, \alpha, E \cup F) = p(A, \alpha, E) + p(A, \alpha, F)$ for all $\alpha \in \mathscr{S}$ (p is a probability function), so that $f_{A,E} + f_{A,F} \leqslant 1$ (recall that $f_{A,E}(\alpha) = p(A, \alpha, E)$ for all $\alpha \in \mathscr{S}$), which means that $\mu_A(E) \perp \mu_A(F)$. Similarly, for $E_1, E_2, \ldots$ with $E_i \cap E_j = \varnothing$ for $i \neq j$ we have $p(A, \alpha, E_1 \cup E_2 \cup \ldots) = p(A, \alpha, E_1) + p(A, \alpha, E_2) + \ldots$ for all $\alpha \in \mathscr{S}$, and consequently $\mu_A(E_1 \cup E_2 \cup \ldots) = \mu_A(E_1) \cup$

$\mu_A(E_2) \cup \ldots$. Thus μ_A is an L-valued measure. Note that it is the first time we have used the assumption that for each $A \in \mathcal{O}$ and each $\alpha \in \mathcal{S}$ the mapping $E \to p(A,\alpha,E)$ is a probability measure on $\mathcal{B}(R)$. The fact that $(L,\leqslant,')$ is an orthocomplemented partially ordered set with a full set of states (probability measures) $\{m_\alpha : \alpha \in \mathcal{S}\}$ is a consequence of the orthogonality postulate only.

The family of L-valued measures $\{\mu_A : A \in \mathcal{O}\}$ is surjective, since every member of L is of the form $f_{A,E}$ for some A and E. We also have

$$m_\alpha \circ \mu_A(E) = m_\alpha(f_{A,E}) = f_{A,E}(\alpha) = p(A,\alpha,E)$$

by the definition of $f_{A,E}$. Hence the first part of the theorem is proved.

To prove the converse, let $(L,\leqslant,')$ be an arbitrary orthocomplemented partially ordered set admitting a full set of probability measures S, and let $\mathcal{O}$ be a surjective set of L-valued measures. Let $p(A,\alpha,E) = \alpha \circ A(E)$ for all $A \in \mathcal{O}$, $\alpha \in \mathcal{S}$, and $E \in \mathcal{B}(R)$. It is evident that p is a probability function. Let $f_1, f_2, \ldots$ be a sequence of pairwise orthogonal experimental functions induced by p where $f_i = f_{A_i,E_i}$. Since $f_i + f_j \leqslant 1$ for $i \neq j$, we have $f_{A_i,E_i}(\alpha) + f_{A_j,E_j}(\alpha) \leqslant 1$ for all $\alpha \in \mathcal{S}$, that is $\alpha \circ A_i(E_i) + \alpha \circ A_j(E_j) \leqslant 1$ for all α. This implies $\alpha \circ A_i(E_i) \leqslant \alpha \circ A_j(R - E_j)$ for all $\alpha \in \mathcal{S}$. Since $\mathcal{S}$ is full, we infer that $A_i(E_i) \leqslant (A_j(E_j))'$ in L. Hence $A_1(E_1) \cup A_2(E_2) \cup \ldots$ exists in L and by the surjectivity of $\mathcal{O}$ $A_1(E_1) \cup A_2(E_2) \cup \ldots = A(E)$ for some $A \in \mathcal{O}$ and $E \in \mathcal{B}(R)$. This implies $\alpha \circ A_1(E_1) + \alpha \circ A_2(E_2) + \ldots = \alpha \circ A(E)$ for all α, that is $\alpha \circ A(R - E) + \alpha \circ A_1(E_1) + \alpha \circ A_2(E_2) + \ldots = 1$ for all $\alpha \in \mathcal{S}$. This means that $g + f_1 + f_2 + \ldots = 1$ where $g(\alpha) = p(A,\alpha, R - E)$ for all $\alpha \in \mathcal{S}$. Hence $f_1, f_2, \ldots$ is orthogonal in the sense of Definition 1 and the orthogonality postulate holds. It is evident that the logic of p is isomorphic to L. This concludes the proof of the theorem.

The theorem shows that the logic of a probability function is an orthocomplemented partially ordered set if and only if the orthogonality postulate holds. This explains why the structure of orthocomplemented partially ordered set is so basic and underlies all known physical theories.

The orthogonality postulate corresponds to Axiom V of Mackey, which turns out to be the most essential in determining the structure of our system. There are other assumptions usually made in axiomatic quantum mechanics as formulated by Mackey, namely: 1^0 $p(A,\alpha,E) = p(A',\alpha,E)$ for all α and all E implies $A = A'$ and similarly for states (Axiom II of Mackey), 2^0 every L-valued measure corresponds to an observable (Axiom VI and implied by it Axiom III), 3^0 the set of all states is closed under taking convex combinations (Axiom IV), and 4^0 for every $f \in L$ different from 0 there is $\alpha \in \mathcal{S}$ such that $m_\alpha(f) = 1$ (Axiom VIII). It is easy to see that even if these postulates do not hold they can be made to hold by introducing equivalence relations in the set of states and in the set of observables (Axiom II) and by suitable extensions of the set of states and the set of observables (the remaining axioms with the possible exception of Axiom VIII, but this one is used only after passing to Hilbert space to show that every unit vector determines a pure state). Hence as far as the abstract structure of axiomatic quantum mechanics is concerned, the orthogonality postulate is the most essential one.

As we have mentioned in the introduction, to pass to the Hilbert space formalism of quantum mechanics we have to assume that only L is isomorphic to the lattice of all closed subspaces of a suitable Hilbert space. This is the so-called Hilbert space axiom of Mackey (Axiom VII). In this formulation this axiom can be given little direct physical motivation. However, as was shown in Mączyński (1972), Mackey's axiom system can be extended to include more assumptions about the probability function so that the Hilbert space axiom follows from postulates which admit more natural physical interpretation.

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